(Section from the book "Computer networks", authors: S.I.Samoilenko, A.A. Davidov, V.V.Zolotarev, E.I.Tretjakova, "Science", Moscow, 1981, p.278)

Error propagation effect in majority decoders.

From the demonstrated above outcomes relating to the main MTD properties follows that the increase of number of correcting attempts for earlier decoded symbols with the help MTD is useful, as at each change of information bits of a code there is a transition to more verisimilar decisions. **However from these results does not follow at all, that MTD will reach necessarily the optimum solution.** For many codes there is a rather large number of such channel error combinations, that they may be corrected by optimum decoder, but are not corrected in a proper way by MTD. It is connected with the fact that threshold decoders are true in a rather large degree are subject to influencing of effect of an error propagation (EP) effect. Second and other sequentially connected threshold switches (TS) in convolution version, or the simply following iterations of correction in a block code in many cases are compelled to work, basically, with flows of error bursts from the previous stages of decoding. For this reason the miscellaneous attempts of many repeated decoding explorers of different codes have not brought significant outcomes.

Below as an example the method of an estimation of an error propagation (EP) effect for self-orthogonal codes (SOC) consisting is reviewed. The estimations of probability of of single errors and packets appearance at the TD output are calculated with the multidimensional probability generators functions (PGF) help. This method is actively used at selection codes in the least degree subjected influencing of a EP, which ones essentially improve the MTD characteristics.

Let the error probability $\mathbf{p_0}$ is given for a binary symmetrical channel (BSC). Let's select from some SOC the first informational symbol and another arbitrary informational symbol $\mathbf{i_m}$, $\mathbf{m}>0$. For each of them it is possible to enumerate those components of a syndrome, which ones will derive set of orthogonal checks and move at decoding at threshold switch (TS) of customary TD. Let's enumerate these checks from 1 up to **J=d-1**. If thus any symbol $s_n \in \overline{S}$ will be given at TS in both cases, it will have two different numbers conforming to symbols $\mathbf{i_0}$ and $\mathbf{i_m}$.

Let's consider now all informational and control symbols included in checks even concerning one of $\mathbf{i_o}$ or $\mathbf{i_m}$. Let it is necessary to construct an upper bound estimation of joint error probability of an customary TD in two symbols **i_o** и **i_m**: $P(\xi_0 = 1, \xi_m = 1)$.

Then, similarly unidimensional case of calculus of an error probability in the first symbol **$P_1(e)$** of customary TD, we get, that for any $i_k, k \ge m$, the error in which one enters as addend in **n**-th check for **i_m** and in l-st check for **i_0**, the probability generator becomes $A_{nl}(x,y) = p_0 x_n y_l + q_0$. If the error in some **i_k** enters in checks only for **i_0** or **i_m**, PGF transforms in a unidimensional form: $A_n(x) = p_0 x_n + q_0$ or $A_l(y) = p_0 y_l + q_0$ accordingly. For control symbols, which one enter as addends only in checks concerning one of **i_0** or **i_m**, unidimensional PGF have the same view. As we considered of random errors in TD in **i_0** and **i_m**, in that (single, because a code - is SOC!) control symbol (if it is), which one corresponds to check participating in the TD decision about **i_0** and **i_m**, and, it is necessary to take into account, that through a feedback from a threshold switch in this check enters an own TD error. PG for this symbol looks like $A_{nl}(x,y) = p_0 x_n + q_0 y_l$.

At last, it is necessary to take into account last rule, which one is used at derivation of an estimation of appearance of two TD errors. It is supposed for all information symbols $i_k, 0 < k < m$, that at their decoding through feedback in the syndrome register S enter the true error values in **i_k**. Therefore errors in these symbols at decoding **i_m** are absent. Such hypothetical TD is called as the decoder with "genie".

In this case description of TD behavior through properties of such special decoder will allow easier to get indispensable estimations for the considered conventional majority decoder.

Let's mark also, that if we consider probability of two errors in an adjacent couple of symbols **i_0** and **i_1**, any additional suppositions about decoding with "genie" are not required.

Let's now enter rule of multiplication multidimensional PG. It is that the exponents at **x_n** and **y_l** with identical indexes must be modulo 2 summarized. Thus, all exponents at **x_n** and **y_l** have only values 0 or 1. It is defined by that the values of checks can be only 0 or 1, and the errors relating identical checks have identical indexes and the modulo 2 sum of even numbers of errors is 0.

As a result of all PGF multiplication we get the sum $A_{0m}(x,y) = \sum a_{n_1 n_2 \ldots n_d, k_1 \ldots k_d} x^{n_1+n_2+\ldots+n_d} y^{k_1+\ldots+k_d}$ of **2^{2d}** addends. The factors **a** …. with indexes, where **n_i** and **k_i** are equal 0 or 1, are probabilities of check values of instituted by these indexes. For example, at **d**=5 a factor **$a_{01000,10000}$** is peer to probability that the first check for **i_0** is equal 1, and remaining are equal 0. Hereinafter we shall consider that the first index in group falls into value of a decoded symbols. So, for example, in the second group of indexes their values indicate that **a….,….** is equal to joint probability of a described set for a symbol **i_0** and channel error **e_m** =1 in **i_m** at exact next checks relatively **i_m**. After obtaining the reviewed factors as a result of summation of that of them, for which sum of indexes in each group is more then **T**, we receive an estimation $P(\xi_0 = 1, \xi_m = 1)$ for TD with "genie", as the sum of factors **a…..** is equal to probability of all possible combinations of errors in symbols **i_o**, **i_m** and checks resulting to errors of decoding.

For example, for a convolutional SOC code with **R**=1/2, generating polynomial by the way of power at non-zero members G = (0,1,4,6) and code distance **d**=5 resultant PGF for **i_o** and **i_1** will be the next:

$$A_{01}(x,y) = (p_0x_0+q_0)(p_0x_1+q_0)(p_0x_2+q_0y_1)(p_0x_2y_0+q_0)(p_0x_3y_4+q_0)\times$$
$$\times(p_0x_3y_3+q_0)(p_0x_3+q_0)(p_0x_4y_2+q_0)(p_0x_4y_3+q_0)(p_0x_4y_4+q_0)\times$$
$$\times(p_0x_4+q_0)(p_0y_2+q_0)(p_0y_3+q_0)(p_0y_4+q_0)(p_0y_4+q_0).$$

Here we also suppose, that for errors in **i_0** and **i_1** the zero indexes are selected, and the checks numbering from 1 up to 4 is executed pursuant to numbering of checks in a check matrix of this code top-down.

After factor summarizing according to the rules, indicated on the previous page, and yardsticks of selection of decomposition values **a…..,….** . it is possible to get, that at **p_0**<<1 $P(\xi_0=1,\xi_1=1)\approx 26p_0^3$, and $P(\xi_0=1,\xi_i=1)\approx 8p_0^3$, if $2\le i\le 4$. At **i**>4 these probabilities already have the order **p_0^4**, and consequently for not so high noise levels their contribution in EP effect is insignificant. Then the upper-bound estimate of weight 2 and more packet probability $P(\xi_0=1,1)$ occurrence is peer at a small noise

$$P(\xi_0=1,1)=\sum_{i=1}^{4}P(\xi_0=1,\xi_i=1)=50p_0^3 \ . \tag{1}$$

As it is possible to test, that the probability of an error in the first symbol of a considered code $P_1(e)=85p_0^3$, the lower estimation of probability of a single error in the first symbol can be submitted by the way

$$P_H(\xi_0=1,0)=P_1(e)-P(\xi_0=1,1)=35p_0^3 \ . \tag{2}$$

As the probability was above computed precisely, the probability of occurrence of a single error estimates from above at a small noise as $P_B(\xi_0=1,0)=P_1(e)-P(\xi_0=1,\xi_1=1)=59p_0^3$. Therefore, the probability of a single error occurrence for TD with "genie" lies in range $(35\div 59)p_0^3$.

Now we shall remark, that when TD with "genie" make an error in no one symbol $i_k, 0<k<k=n_AR$, within the constraint code length **n_A**, TD nor commits any error, as the "genie" decisions coincide with the solution of TS. But just for such channel errors sequences the estimation (2) was obtained. Means, it is fair and for customary TD.

Capability to estimate probability of two errors occurrence within the limits of **n_A** allows to extend this method on errors decoding packets of any weight. Thus has appeared, that for maintenance of high MTD performance, instituted by EP, it is enough to discuss packets of weight 3. In this case it is necessary to make calculations in variable space of dimension **2^{3d}**. But at the codes analysis with **d**≥7 and more this problem is too difficult for direct calculations.

Special methods of considerable calculations simplification of packets appearance probabilities were therefore designed, which one then have allowed to formulate a complex of approximately 20 yardsticks, with which one it is necessary to plot lengthy codes with very small probabilities of decoding error bursts appearance. The corresponding programs at construction of such codes with length **n** about $\mathbf{n}^4$ operations, that allows to plot fast enough very effective codes up to lengths about **n**=1'000' 000 bits.

As at decoding near to channel capacity an application of very lengthy codes is only possible, the development of described above methods of codes creation completely has solved problem of codes selection for high-performance decoders of MTD class for channels with a large noise.

Thus, simultaneous solution of TD advancing problem, which one has transformed into MTD, executing process of global extremum searching on a very large set of integer variables, and also exhaustive solution of the EP effect analysis problem, that has resulted in construction absolutely new codes with a small level of EP effect, allow to construct new elementary, but very effective algorithm of decoding. As the very potent method of error correction follows from experimental outcomes on effective decoder simulation for specially constructed codes, MTD - is a very capable method, which one able almost always to find the optimum solutions at rather high levels of a channel noise.

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